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Dynamical Equations of Nonrigid Satellites

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Nomenclature

dV	= differential volume element
$\tilde{\mathbf{F}}$	= force per unit mass
$\mathbf{F}$	= total external force on satellite
$\mathbf{G}^m$	= total torque on m about its c.m.
$\mathbf{H}^m$	= angular momentum of m about its c.m.
$[\mathbf{J}^m]$	= inertia dyadic of m about its c.m.
M	= mass of entire satellite; also indicates the entire satellite
M^m	= mass of satellite main body
m	= satellite main body
o	= origin of body frame b
$\mathbf{R},\mathbf{R}^m$	= vector from o to c.m. of M and m, respectively
$\mathbf{r}$	= vector from o to dV
$\mathbf{V}$	= absolute translational velocity of the c.m. of M
$\mathbf{y},\mathbf{y}^m$	= vector to dV from c.m. of M and m, respectively
$\mathbf{Z}$	= absolute acceleration of $\mathbf{R}$ with $\ddot{q}_j$ and $\dot{\boldsymbol{\omega}}$ components deleted
ρ	= density
$\boldsymbol{\omega}$	= absolute angular velocity of frame b
[ ]	= dyadic

Introduction

RECENTLY published techniques enable dynamical equations of deformable satellites to be derived directly and efficiently when the satellite is represented as a set of rigid bodies joined at hinge points in a tree topology.[1,2] The purpose of the present Note is to indicate an analogous method which is applicable when satellite nonrigidity is modeled by generalized coordinates. The approach enables a direct development of dynamical equations that can include configuration changes and large deformations, angular rates, and attitude motions. The dynamical equations of such systems commonly are derived by a formal application of Lagrange's equation. The study by Farrell and Newton[3] is an excellent example of the formal use of Lagrange's equation in a difficult nonrigid satellite problem. The Lagrangian approach, however, has several disadvantages. The manual labor needed to derive and differentiate the kinetic energy expression can be time consuming and difficult to accomplish without error. The resulting equations are difficult to modify after they have been developed, and the significance of individual terms often is obscure. In satellite problems, the resulting rotation equations often are less convenient than those obtained by angular momentum techniques. The generalized forces of rotation are not merely torque vector components resolved on an orthogonal frame, and the possibility of simplifying the rotation equations by using internal torques is precluded. Assuming that attitude is specified by Euler angles, the dynamics equations are made cumbersome by the presence of trigonometric functions of these angles.

The aforementioned disadvantages in the use of Lagrange's equation are alleviated by the technique described here. Two main equations are presented. The first defines the satellite dynamics relative to a frame b which is attached to the satellite in an arbitrary manner. The second defines frame b's rotational kinetics. Translation of the c.m. is not treated, because the equations are well known. The relative motion equation essentially is equivalent to an equation given by Frazer, Duncan, and Collar,[4] whereas the rotation equation essentially is equivalent to a result attributed to Liouville.[5] The advantage of the equations of the present Note over those which have appeared previously is that the individual terms have been expressed as explicitly as possible without a sacrifice in generality using generalized coordinates and their derivatives. This enables dynamical equations of nonrigid satellites to be developed in an efficient manner with a reduced probability of algebraic or conceptual errors.

Equations used by Buckens[6] and Ashley[7] are special cases of those derived here. Both these authors employed normal modes and a linearized deformation model. Neither was specifically concerned with developing general nonlinear dynamical equations of nonrigid satellites. The "hybrid coordinate" approach used by Likins and Wirsching[8] is a special case of the method of separating system dynamics into relative and rotational degrees of freedom employed in the present work. Reference 8 was concerned mainly with the efficiency of the linearized dynamical equations when used in a computer simulation; emphasis was on simplifying the inertia matrix. The present study does give some consideration to the inertia matrix problem. The main interest, however, is on reducing the effort needed to derive nonlinear equations.

The final equations are Eqs. (3–7, and 10–11). The variables q, which define the satellite state relative to frame b, have been separated into two sets: 1) a set indicated by subscripts i and $j(i,j = 1 \text{ to } n)$ whose dynamics are obtained from the kinetics equations and 2) a set indicated by subscript $\alpha(\alpha = n + 1 \text{ to } N)$ whose time responses are known a priori or established by a control law. Subscripts u and $v(u,v = 1 \text{ to } N)$ indicate the complete q set. Use of the same subscript twice in a product of terms indicates a summation over the range of the indexes. A comma is used in the conventional manner to indicate differentiation. Dyadic brackets [ ] around a vector indicate the usual skew-symmetric form.

In the rotation equations, the satellite is represented as a main body m to which auxilliary bodies s are attached. Neither m nor s are assumed to be rigid. s consists of those bodies whose interaction torques $\mathbf{G}^s$ and forces $\mathbf{F}^s$ on m are readily computable. When applicable, the s technique has the potentiality of simplifying the rotation equations. In some problems, certain components of $\mathbf{F}^s$ and $\mathbf{G}^s$ can be obtained only from the external loading, including the inertial forces, on s. An example is a satellite with a main body m and booms s whose bending deformations are defined by

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kinetics equations but whose torsional deformations are omitted. Superscript m in the equations indicates parameters defined only over m. Parameters without the m superscript are defined over the entire satellite.

When applying the method, the steps are as follows: 1) define a model of the satellite nonrigidity using variables q_u; 2) if advantageous, modify or simplify the equations of this Note to fit the requirements of the problem; 3) develop numerical values of, equations for, or a technique of otherwise handling the volume integrals in the equations; and 4) derive any additional miscellaneous equations that may be needed. The formal use of Lagrange's equation requires steps 1, 3, and 4 plus deriving and differentiating a kinetic energy expression. In the present method, step 3 comprises the major part of the effort; in general, it is necessary to use a nonrigidity model that permits q_i to be moved outside the integrands. When feasible, it is advantageous to locate the origin o of frame b at the c.m. of either M or m. The location of o was unspecified in the present work in order to permit maximum freedom in selecting q_u. If the equations are employed in a computer simulation, any of the standard approaches can be used when integrating $\dot{\boldsymbol{\omega}}$ to get attitude. Maximum flexibility is obtained by converting attitude immediately into direction cosines and employing direction cosines explicitly elsewhere in the simulation.

Relative Motion Equations

Satellite kinetic energy T can be expressed as follows:

$$T = 0.5M\mathbf{V}\cdot\mathbf{V} + 0.5\int_M \rho\dot{\mathbf{y}}\cdot\dot{\mathbf{y}}dV + \boldsymbol{\omega}\cdot\int_M \rho\mathbf{y}\times\dot{\mathbf{y}}dV - 0.5\boldsymbol{\omega}\cdot\int_M \rho[\mathbf{y}][\mathbf{y}]dV\cdot\boldsymbol{\omega} \quad (1)$$

In the present work, the vector time derivatives are defined relative to frame b. All vectors and dyadics, except V, are assumed to be resolved on frame b. The vector $\mathbf{y}$ then is a function solely of q_u, while $\dot{\mathbf{y}}$ is a function solely of q_u and $\dot{q}_u$. The desired relative motion dynamics equation is obtained by substituting Eq. (1) into Lagrange's equation, performing the differentiations, and rearranging. The resulting "coefficients" then can be transformed from $\mathbf{y}$ to $\mathbf{r}$ and $\mathbf{R}$ variables by using the relations

$$\mathbf{y} = \mathbf{r} - \mathbf{R} \quad (2)$$

$$\mathbf{R} = (1/M)\int_M \rho\mathbf{r}dV \quad (3)$$

The final equations are as follows

$$A_{ij}{}^1\ddot{q}_j - \mathbf{A}_i{}^2\cdot\dot{\boldsymbol{\omega}} = Q_i - A_{i\alpha}{}^1\ddot{q}_\alpha + 2\mathbf{A}_{iu}{}^3\cdot\boldsymbol{\omega}\dot{q}_u - A_{iuv}{}^4\dot{q}_u\dot{q}_v - \boldsymbol{\omega}\cdot[\mathbf{A}_i{}^5]\cdot\boldsymbol{\omega} \quad (4)$$

where

$$Q_i = \int_M \rho\mathbf{r}_{,i}\cdot\tilde{\mathbf{F}}dV - \mathbf{R}_{,i}\cdot\mathbf{F} \quad (5)$$

$$A_{iu}{}^1 = \int_M \rho\mathbf{r}_{,i}\cdot\mathbf{r}_{,u}dV - M\mathbf{R}_{,i}\cdot\mathbf{R}_{,u} \quad (6a)$$

$$\mathbf{A}_i{}^2 = \int_M \rho\mathbf{r}_{,i}\times\mathbf{r}dV - M\mathbf{R}_{,i}\times\mathbf{R} \quad (6b)$$

$$\mathbf{A}_{iu}{}^3 = \int_M \rho\mathbf{r}_{,i}\times\mathbf{r}_{,u}dV - M\mathbf{R}_{,i}\times\mathbf{R}_{,u} \quad (6c)$$

$$A_{iuv}{}^4 = \int_M \rho\mathbf{r}_{,i}\cdot\mathbf{r}_{,uv}dV - M\mathbf{R}_{,i}\cdot\mathbf{R}_{,uv} \quad (6d)$$

$$[\mathbf{A}_i{}^5] = \int_M \rho[\mathbf{r}][\mathbf{r}_{,i}]dV - M[\mathbf{R}][\mathbf{R}_{,i}] \quad (6e)$$

$\tilde{\mathbf{F}}$ in Eq. (5) includes both external and internal forces. An obvious modification to the integral is needed when including point or surface forces. When developing the generalized gravity-gradient force, a series expansion of $\tilde{\mathbf{F}}$ about the system c.m. is needed; the large zero-order component then will cancel out in Eq. (5). A modification to Eqs. (4) and (6), which can be useful in computer simulations, is obtained by deleting the $\mathbf{R}_{,i}$ terms from the A coefficients on the right side of Eq. (4). The quantity $M\mathbf{R}_{,i}\cdot\mathbf{Z}$ then must be added to the right side of Eq. (4) where

$$\mathbf{Z} = 2\boldsymbol{\omega}\times\mathbf{R}_{,u}\dot{q}_u + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}) + \mathbf{R}_{,uv}\dot{q}_u\dot{q}_v + \mathbf{R}_{,\alpha}\ddot{q}_\alpha \quad (7)$$

When applying Eqs. (4–6) to a nonrigid spinning satellite, it is not necessary to add a centrifugal potential function[9] to Q_i. This function is supplied by the $\mathbf{A}_i{}^5$ term in Eq. (4). However, both the first- and second-order q_u terms must be included in the deformation model $\mathbf{r}(q_u)$.

Rotation Equations

Angular momentum principles provide the simplest approach for deriving the frame b rotational kinetics equations. The starting equations are the following

$$\mathbf{H}^m = \int_m \rho\mathbf{y}^m\times\dot{\mathbf{y}}^m dV - \int_m \rho[\mathbf{y}^m][\mathbf{y}^m]dV\cdot\boldsymbol{\omega} \quad (8)$$

$$\mathbf{G}^m = \dot{\mathbf{H}}^m + \boldsymbol{\omega}\times\mathbf{H}^m \quad (9)$$

$\mathbf{G}^m$ includes the interaction forces and torques exerted on m by s. The desired rotational equation is obtained by differentiating Eq. (8), substituting into Eq. (9), and rearranging. The resulting "coefficients" then can be transformed from $\mathbf{y}^m$ to $\mathbf{r}^m$ and $\mathbf{R}^m$ variables by using equations for m similar to Eqs. (2) and (3). The result is as follows:

$$-\mathbf{A}_j{}^{m2}\ddot{q}_j + [\mathbf{J}^m]\cdot\dot{\boldsymbol{\omega}} = \mathbf{G}^m + \mathbf{A}_\alpha{}^{m2}\ddot{q}_\alpha + 2[\mathbf{A}_u{}^{m5}]\cdot\boldsymbol{\omega}\dot{q}_u - \mathbf{A}_{uv}{}^{m6}\dot{q}_u\dot{q}_v - [\boldsymbol{\omega}][\mathbf{J}^m]\cdot\boldsymbol{\omega} \quad (10)$$

where

$$[\mathbf{J}^m] = -\int_m \rho[\mathbf{r}][\mathbf{r}]dV + M^m[\mathbf{R}^m][\mathbf{R}^m]$$

$$\mathbf{A}_{uv}{}^{m6} = \int_m \rho\mathbf{r}\times\mathbf{r}_{,uv}dV - M^m\mathbf{R}^m\times\mathbf{R}_{,uv}{}^m \quad (11)$$

The equations for $\mathbf{A}^{m2}$, $\mathbf{A}^{m5}$, and $\mathbf{R}^m$ are identical to those given in Eqs. (3) and (6) except that M and $\mathbf{R}$ are replaced by M^m and $\mathbf{R}^m$ and the integrations are performed only over m. If $\mathbf{G}^m$ includes inertial forces, a rearrangement of Eq. (10) is needed to group together all the $\dot{\boldsymbol{\omega}}$ and $\ddot{q}_j$ terms. An alternate form of Eq. (10) is obtained by deleting the $\mathbf{R}^m$ terms from the coefficients on the right side and adding the single term $M^m\mathbf{R}^m\times\mathbf{Z}^m.\mathbf{Z}^m$ is defined by Eq. (7) with the $\mathbf{R}$ terms replaced by $\mathbf{R}^m$.

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